

THE FROBENIUS THEOREM FOR \mathbb{Z}_2^n -SUPERMANIFOLDS

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ABSTRACT. We continue the development of \mathbb{Z}_2^n -supergeometry, a natural generalization of classical (\mathbb{Z}_2 -graded) supergeometry, by proving the Frobenius theorem for integrable distributions on differentiable \mathbb{Z}_2^n -supermanifolds. Both the local and global versions of the theorem are addressed.

1. INTRODUCTION

Recently, motivated by various physical applications including those of string theory [1] and parastatistics [14], there has been work on the foundations of the theory of differentiable \mathbb{Z}_2^n -supermanifolds. A Berezin-Leites style ringed space definition of \mathbb{Z}_2^n -supermanifold was given in [4], and a proof of the \mathbb{Z}_2^n -graded analogue of the celebrated theorem of Batchelor [2] [3] and Gawedzki [10] in [5]. The basic results of differential calculus on \mathbb{Z}_2^n -supermanifolds were proven in [6]. The present paper continues this development of the foundations of \mathbb{Z}_2^n -supergeometry.

The classical Frobenius theorem is a key tool in the classical theory of differentiable manifolds, allowing one to build a submanifold of a manifold from knowledge of its tangent bundle. It is therefore desirable to extend this theorem to \mathbb{Z}_2^n -supermanifolds. The extension for $n = 1$ is by now well-known; proofs may be found in e.g. [9], [13]. The goal of the present paper is to formulate and prove the Frobenius theorem for differentiable \mathbb{Z}_2^n -supermanifolds.

We now explain the structure of the paper. Section 3 is dedicated to the preliminaries required, largely developing some basic linear algebra in the \mathbb{Z}_2^n -graded context, as well as the generalization of Nakayama's lemma to this context. The crucial property of \mathcal{I} -adic Hausdorff completeness of the structure sheaf of a \mathbb{Z}_2^n -supermanifold is required to ensure that this linear algebra is well-behaved. Section 4 is dedicated to a discussion of distributions on \mathbb{Z}_2^n -supermanifolds. Here the \mathcal{I} -adic topology on the structure sheaf comes into play. In section 5, we prove the local version of the Frobenius theorem. Here we have the novel phenomenon, not present in $n = 1$ -supergeometry, of vector fields that are even but not of degree zero, and we analyze the local structure of such vector fields and show that there exists a local coordinate system in which they take a standard form. We then briefly discuss the concepts of embedded and immersed submanifold of a \mathbb{Z}_2^n -supermanifold. In Section 7, we prove the global version of the \mathbb{Z}_2^n -super Frobenius theorem that states that a \mathbb{Z}_2^n -supermanifold is foliated by the integral subsupermanifolds associated to a distribution.

The generalization of the results of the theory of Lie algebras, Lie groups and their representations to the \mathbb{Z}_2^n -graded category is largely open; however, a discussion of \mathbb{Z}_2^n -graded ("color") Lie algebras may be found in [11]. The \mathbb{Z}_2^n -super Frobenius theorem opens the way to a development of the theory of \mathbb{Z}_2^n -super Lie groups and their actions on \mathbb{Z}_2^n -supermanifolds, in particular homogeneous \mathbb{Z}_2^n -superspaces. It would be interesting to address these in future work.

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3. PRELIMINARIES

We will need the following basic criterion for the invertibility of a square matrix with entries in a commutative \mathbb{Z}_2^n -superalgebra.

Lemma 3.1. *Let R be a \mathbb{Z}_2^n -supercommutative ring, J the homogeneous ideal generated by the elements of nonzero degree. Suppose that R is J -adically Hausdorff complete. Let T be an $n \times n$ matrix with entries in R . Then T is invertible if and only if it is invertible mod J .*

Proof. If T is invertible, then clearly $\bar{T} := T \pmod{J}$ is invertible. Conversely, suppose that \bar{T} is invertible, i.e. there is some matrix S such that $ST = I + X$, where X has all entries in J . Hence it suffices to show that any matrix of the form $I + X$, where X has all entries in J , is invertible.

The \mathbb{Z}_2^n -graded associative ring $N := M(n, R)$ of all $n \times n$ matrices with entries in R is an R -module by scalar multiplication. As R is J -adically Hausdorff complete, the same is true of N . (This follows from the fact that N is a free R -module of finite rank, see [6]).

The matrix $I + X$ has a formal inverse given by the geometric series: $(I + X)^{-1} = \sum_{k=0}^{\infty} (-1)^k X^k$. As $X^k \in M(n, J^k)$, the partial sums $\sum_{k=0}^n (-1)^k X^k$ form a J -adically Cauchy sequence in N . By Hausdorff completeness of N , the geometric series converges to a unique limit in N , which is the inverse of $I + X$. \square

The following graded version of Nakayama's lemma will be a key tool in the sequel.

Lemma 3.2 (\mathbb{Z}_2^n -graded Nakayama's lemma). *Let A be a \mathbb{Z}_2^n -supercommutative local ring with maximal homogeneous ideal \mathfrak{m} , E a finitely generated module for A , regarded as an ungraded ring. Let J be the ideal generated by the elements of nonzero degree, and suppose A is J -adically Hausdorff complete. Then:*

- (1) *If $\mathfrak{m}E = E$, then $E = 0$. More generally, if H is a submodule of E such that $E = \mathfrak{m}E + H$, then $E = H$.*
- (2) *Let $\{v_i\}_{1 \leq i \leq p}$ be a basis for the ungraded k -vector space $E/\mathfrak{m}E$, where $k := A/\mathfrak{m}$. Suppose $e_i \in E$ lie above v_i . Then the e_i generate the A -module E . If E is a graded module for A and the v_i homogeneous, then we may choose the e_i to be homogeneous of the same parity as the v_i .*
- (3) *Suppose E is a projective A -module, i.e. $E \oplus F = A^N$ for some A -module F . Then E is free, and the e_i of 2) form a basis of E .*

Proof. The proof is similar to that of the corresponding lemma in [13]. We first remark that if B is a commutative local ring with maximal ideal \mathfrak{n} , then a square matrix R with coefficients in B is invertible if and only if its reduction mod \mathfrak{n} is; indeed, if this is so, $\det(R) \notin \mathfrak{n}$, hence is a unit in B . Now we prove 1). Let $\{e_i\}$ generate E . Since $E = \mathfrak{m}E$, we may write $e_i = \sum_j m_{ij} e_j$ for some $m_{ij} \in \mathfrak{m}$. Letting L be the matrix with entries $\delta_{ij} - m_{ij}$, we have:

$$L \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_p \end{pmatrix} = 0.$$

To prove 1), it suffices to show L has a left inverse. We will prove that L is actually invertible. To see this, let $B = A/J$. The units of A are precisely the elements of $A \setminus \mathfrak{m}$, so $J \subseteq \mathfrak{m}$ by Lem. 2.1 (applied to 1×1 matrices), hence we have homomorphisms:

$$A \rightarrow B = A/J \rightarrow k = A/\mathfrak{m}.$$

Let L_B (resp. L_k) be the reduction of L mod J (resp. mod \mathfrak{m}). B is a commutative local ring with maximal ideal \mathfrak{m}/J , and L_k is the reduction of L_B mod (\mathfrak{m}/J) . But L_k is the identity, so the remark above implies L_B is invertible. But by Lem. 3.1, this implies L is invertible. For the more general statement, suppose $E = H + \mathfrak{m}E$. Then $E/H = \mathfrak{m}(E/H)$, so $E/H = 0$ by what we proved.

To prove part 2), we set H equal to the submodule generated by the e_i . Then $E = \mathfrak{m}E + H$, whence $E = H$ by what we proved above.

In order to prove part 3), first note that F is finitely generated, and that $k = A^N/\mathfrak{m}^N = E/\mathfrak{m}E \oplus F/\mathfrak{m}F$. Let (w_j) be a basis of $F/\mathfrak{m}F$ and let f_j be elements of F lying over the w_j . By 2), the e_i, f_j generate A^N , and the e_i (resp. f_j) generate E (resp. F). Let X denote the $N \times N$ matrix whose columns are the coordinate vectors of e_1, \dots, f_1, \dots in the standard basis of A^N . Then as the e_i, f_j form a basis, we have $XY = I$ for some N by N matrix Y with entries in A . Reducing mod J , we have $X_B Y_B = I$. But as B is commutative, we have $Y_B X_B = I$ as well. By Prop. 3.1, X has a left inverse over A , which must be Y . Suppose there is a linear relation between the e_i and f_j , and let x be the column vector whose components are the coefficients of this relation. Then $Xx = 0$, but then $x = YXx = Xx = 0$. Hence E is a free module with basis e_i . \square

Remark. It would be interesting to see if it is possible to remove the assumption that the ring A is J -adically Hausdorff complete in the statement of the graded Nakayama's lemma. This is not necessary for our purposes here, as we work exclusively with the local rings of germs of functions on a \mathbb{Z}_2^n -supermanifold, which are indeed \mathcal{J} -adically Hausdorff complete. However, for other applications (e.g. the development of algebraic \mathbb{Z}_2^n -supergeometry), one might potentially have to consider rings for which this hypothesis is not satisfied, and it would be desirable to remove it.

4. DISTRIBUTIONS ON \mathbb{Z}_2^n -SUPERMANIFOLDS

Definition 1. Let M be a \mathbb{Z}_2^n -supermanifold. A *distribution* on M is a graded subsheaf \mathcal{D} of the tangent sheaf $\mathcal{T}M$ which is locally a direct factor, i.e. for any point $m \in |M|$ there exists an open neighborhood $U \ni m$ and an $\mathcal{O}(U)$ -module \mathcal{D}' such that $\mathcal{T}_n M = \mathcal{D}_n \oplus \mathcal{D}'_n$ for all $n \in U$. We consider \mathcal{D} as a sheaf of topological modules, endowed with the \mathcal{J} -adic topology from \mathcal{O} .

The following will allow us to define the key concept of the *rank* of a distribution.

Lemma 4.1. Let $m \in M$ and let $\{X_i, \chi_\rho\}$ be vector fields defined in a neighborhood of m such that their associated tangent vectors at m are linearly independent in $T_m M$. Then their germs $\{[X_i]_m, [\chi_\rho]_m\}$ are \mathcal{O}_m -linearly independent in $[\mathcal{T}M]_m$.

Proof. Choose vector fields Y_1, \dots, Y_l such that the tangent vectors $\{(X_i)_m, (\chi_\rho)_m\} \cup \{(Y_1)_m, \dots, (Y_l)_m\}$ form a basis of $T_m M$. By the \mathbb{Z}_2^n -super Nakayama's lemma, the germs $\{[Y_1]_m, \dots, [Y_l]_m\} \cup \{[X_i]_m, [\chi_\rho]_m\}$ at m form an \mathcal{O}_m -basis of $[\mathcal{T}M]_m$. In particular, $\{[X_i]_m, \dots, [\chi_\rho]_m\}$ are \mathcal{O}_m -linearly independent. \square

Proposition 4.2. Let \mathcal{D} be a distribution. Then \mathcal{D} is a locally free sheaf of \mathcal{O}_M -modules. Furthermore, if $|M|$ is connected, $\dim(\mathcal{D}_m/\mathfrak{m}_m \mathcal{D}_m)$ is independent of m .

Proof. Let $m \in M$. Then the super Nakayama's lemma (Lem. 3.2) implies that \mathcal{D}_m is a free \mathcal{O}_m -module, hence that $\mathcal{D}(U)$ is a free $\mathcal{O}(U)$ -module in a neighborhood U of m . As \mathcal{D} is a distribution, there exists $D \subseteq \mathcal{T}M(U)$ such that $[\mathcal{T}M]_n = [\mathcal{D}]_n \oplus [D]_n$ for all $n \in U$.

Let $\{X_i, \chi_\rho\}$ be vector fields defined on U such that their associated tangent vectors at m are a homogeneous basis for $\mathcal{D}_m/\mathfrak{m}_m\mathcal{D}_m \subseteq T_mM$, and let Y_j, Θ_σ be vector fields defined on U such that their associated tangent vectors at m are a homogeneous basis for $\mathcal{D}_m/\mathfrak{m}_m\mathcal{D}_m$. The associated tangent vectors of $\{X_i, \chi_\rho, Y_j, \Theta_\sigma\}$ at n form a homogeneous basis of T_nM for any $n \in U$. By Lem 4.1, $\{X_i, \chi_\rho, Y_j, \Theta_\sigma\}$ form an \mathcal{O}_n -basis of $[\mathcal{T}M]_n$ for every n . To see that $\{X_i, \chi_\rho\}$ actually generate \mathcal{D} , set $\mathcal{D}'_n := \text{span}_{\mathcal{O}_n}\{[X_i]_n, [\chi_\rho]_n\} \subseteq [\mathcal{D}]_n$, and $D'_n := \text{span}_{\mathcal{O}_n}\{[Y_j]_n, [\Theta_\sigma]_n\}$. Then $[\mathcal{D}'_n]_n \subseteq [\mathcal{D}]_n$ and $[D'_n]_n \subseteq [D]_n$, but as $[\mathcal{D}'_n]_n \oplus [D'_n]_n = [\mathcal{T}M]_n = [\mathcal{D}]_n \oplus [D]_n$, the generation is clear.

It follows from this that $\dim(\mathcal{D}_m/\mathfrak{m}_m\mathcal{D}_m)$ is independent from $m \in M$, if M is connected. \square

Definition 2. Let \mathcal{D} be a distribution on M . The *rank* of \mathcal{D} is the graded dimension

$$\text{rk}(\mathcal{D}) := \dim(\mathcal{D}_m/\mathfrak{m}_m\mathcal{D}_m),$$

where m is any point of M .

Prop 4.2 shows that the rank is well-defined. For a distribution \mathcal{D} on M and a point $m \in |M|$, the real \mathbb{Z}_2^n -super vector space $\mathcal{D}_m/\mathfrak{m}_m\mathcal{D}_m$ will often be denoted by D_m ; geometrically, this is the fiber of the distribution at the point m .

Since the tangent sheaf is a sheaf of topological modules in \mathbb{Z}_2^n -supergeometry, the topological properties of distributions come into play. As is well-known from commutative algebra, if R is a topological ring endowed with the I -adic topology defined by some ideal I , and N a submodule of an R -module M , the I -adic topology on N does not agree in general with the subspace topology induced on N by the I -adic topology of M . However, in our case it is easy to see that the two coincide:

Proposition 4.3. *Let \mathcal{D} be a distribution. Then \mathcal{D} , endowed with the \mathcal{I} -adic topology, is a topological subsheaf of $\mathcal{T}M$, i.e. the inclusion morphism of sheaves of \mathcal{O} -modules $\mathcal{D} \hookrightarrow \mathcal{T}M$ is a topological embedding. Furthermore, \mathcal{D} is \mathcal{I} -adically Hausdorff complete.*

Proof. Let m be any point of M . As \mathcal{D} is a distribution, there exists an open subset $U \ni m$ and a submodule \mathcal{D}' of $\mathcal{T}M(U)$ such that $\mathcal{T}_mM = \mathcal{D}_m \oplus \mathcal{D}'_m$. This implies $\mathcal{I}_m^k \cdot \mathcal{T}_mM = \mathcal{I}_m^k \cdot \mathcal{D}_m \oplus \mathcal{I}_m^k \cdot \mathcal{D}'_m$. From this, it is easy to see that $(\mathcal{I}_m^k \cdot \mathcal{T}_mM) \cap \mathcal{D}_m = \mathcal{I}_m^k \cdot \mathcal{D}_m$. For an arbitrary sheaf of \mathcal{O} -modules \mathcal{F} , let $\mathcal{I}^k \cdot \mathcal{F}$ denote the sheafification of the presheaf $U \mapsto \mathcal{I}^k(U) \cdot \mathcal{F}(U)$. We have an obvious sheaf morphism $\mathcal{I}^k \cdot \mathcal{D} \hookrightarrow (\mathcal{I}^k \cdot \mathcal{T}M) \cap \mathcal{D}$ given by inclusion. Then the preceding discussion implies that $\mathcal{I}^k \cdot \mathcal{D} = (\mathcal{I}^k \cdot \mathcal{T}M) \cap \mathcal{D}$, which amounts to saying that the inclusion $\mathcal{D} \hookrightarrow \mathcal{T}M$ is a topological embedding. That \mathcal{D} is \mathcal{I} -adically Hausdorff complete follows from the fact that it is a locally free sheaf of \mathcal{O} -modules and the Hausdorff completeness of \mathcal{O} . \square

We conclude this section by giving the appropriate generalizations of the notions of involutive and integrable distributions to \mathbb{Z}_2^n -supergeometry. Recall that the \mathbb{Z}_2^n -super Lie bracket on vector fields is:

$$[X, Y](f) := X(Yf) - (-1)^{\langle \deg X, \deg Y \rangle} Y(Xf).$$

Definition 3. A distribution \mathcal{D} is *involutive* if and only if \mathcal{D} is closed under \mathbb{Z}_2^n -graded Lie bracket, i.e. if X, Y are vector fields in \mathcal{D} defined near m , $[X, Y]$ lies in \mathcal{D} .

Definition 4. A distribution \mathcal{D} is *integrable* if and only if for every point $m \in M$, there exists a coordinate system (x, ξ) centered at m such that \mathcal{D}_m is spanned by $\{\partial/\partial x^i, \partial/\partial \xi^j\}_{1 \leq i \leq r, j \in s}$.

5. LOCAL \mathbb{Z}_2^n -SUPER FROBENIUS THEOREM

We have the following analogue of the classical theorem of Frobenius characterizing integrable distributions.

Theorem 5.1 (Local Frobenius theorem on \mathbb{Z}_2^n -supermanifolds). *A distribution \mathcal{D} on a \mathbb{Z}_2^n -supermanifold M is integrable if and only if \mathcal{D} is involutive.*

That an integrable distribution is involutive is obvious, since coordinate vector fields always \mathbb{Z}_2^n -supercommute. The converse will occupy the rest of the proof. We begin by proving a series of auxiliary results on the local structure of distributions.

Proposition 5.2. *Let \mathcal{D} be an involutive distribution. Then \mathcal{D} has a local basis of supercommuting vector fields in a neighborhood of any point.*

Proof. Let m be a point of M and (x, θ) be a coordinate system near m in some open set U . Let $r|\mathbf{s}$ be the rank of \mathcal{D} . Let X_i , $1 \leq i \leq r$ (resp. χ_σ , $1 \leq \sigma \leq s$) be degree zero (resp. nonzero-degree) vector fields whose germs at m form a basis of \mathcal{D}_m . The coefficients of the X_i and χ_σ in the basis $\partial/\partial x^j, \partial/\partial \theta^\rho$ form an $r|\mathbf{s} \times p|\mathbf{q}$ matrix T having the form

$$T = \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix}$$

where a (resp. b) is an $r \times p$ (resp. $\mathbf{s} \times \mathbf{q}$) matrix of degree zero functions, and α (resp. β) is an $r \times \mathbf{q}$ (resp. $(\mathbf{s} \times p)$) matrix of nonzero degree functions. The matrix T must have rank $r|\mathbf{s}$ by the linear independence of the X_i and χ_σ .

By renumbering coordinates (without regard for their degree), we may assume the left $r|\mathbf{s}$ square block submatrix T_0 of T is an invertible degree zero matrix. These permutations may destroy the standard ordering of the columns of T as well as the coarse \mathbb{Z}_2 -grading of T by parity; however, note such permutations only switch the columns of T and hence preserve the degrees of the rows of T .

As $\deg(T_0) = 0$, left multiplication of T by T_0 preserves the degree of the rows of T . Therefore, we have that $T_0^{-1}T$ is a matrix of the form:

$$\begin{pmatrix} I_r & 0 & * \\ 0 & I_s & * \end{pmatrix}.$$

Since T_0 is invertible, the span of the rows of $T_0^{-1}T$ is the same as the span of the rows of T ; the degrees of the rows of T have also been preserved by all operations performed on T . We may therefore assume that for some (not necessarily standard) ordering of the θ^σ s, the X_i and χ_σ are of the form:

$$(5.1) \quad X_i = \frac{\partial}{\partial x^i} + \sum_{j>r} a_{ji} \frac{\partial}{\partial x^j} + \sum_{\rho>s} \alpha_{j\rho} \frac{\partial}{\partial \theta^\rho}$$

$$(5.2) \quad \chi_\sigma = \frac{\partial}{\partial \theta^\sigma} + \sum_{i>r} \beta_{ji} \frac{\partial}{\partial x^i} + \sum_{\rho>s} b_{j\rho} \frac{\partial}{\partial \theta^\rho}$$

By the hypothesis that \mathcal{D} is involutive, the supercommutators $[X_i, X_j]$ lie in \mathcal{D} , hence $[X_i, X_j] = \sum_{t \leq r} f_t X_t + \sum_{\tau \leq s} \varphi_\tau \chi_\tau$. However, one sees from (5.1) that $[X_i, X_j]$ is an $\mathcal{O}(U)$ -linear combination of the $\partial/\partial x^k$ for $k > r$ and the $\partial/\partial \theta^\rho$ for $\rho > s$. It follows that all f_t, φ_τ are zero, i.e. the X_j all supercommute with each other. The cases of $[X_i, \chi_\sigma]$ and $[\chi_\rho, \chi_\sigma]$ are completely analogous. \square

We will now show that given a degree 0 local vector field which is non-degenerate at a point, there exists a local coordinate system in which the given vector field takes a canonical form. We will need the following simple lemma.

Lemma 5.3. *Let U be an open subset of a \mathbb{Z}_2^n -supermanifold, and suppose $\{f_k\}$ converges \mathcal{J} -adically in $\mathcal{O}(U)$. Then*

$$[\widetilde{\lim_{k \rightarrow \infty} f_k}](x) = \lim_{k \rightarrow \infty} \tilde{f}_k(x).$$

Proof. Without loss of generality, we may assume $f_k \rightarrow 0$ \mathcal{J} -adically in $\mathcal{O}(U)$. Then there is some M such that $f_k \in \mathcal{J}$ for all $k \geq M$. Since the reduction of any function in \mathcal{J} is zero, $0 = \tilde{f}_k$ for all $k \geq M$, which implies the lemma. \square

Lemma 5.4. *Let X be a degree zero vector field defined in a neighborhood of m such that X_m is a nonzero tangent vector. Then there is a coordinate system (x, ξ) centered at m such that $X = \partial/\partial x^1$.*

Proof. As the result is purely local, we may assume M is some open subset U of $\mathbb{R}^{p|q}$, with coordinates (z, ζ) centered at $m = 0$. We note that this implies $\mathcal{T}M(U)$ is a free $\mathcal{O}(U)$ -module. Since X has degree 0, it leaves the ideal \mathcal{J} invariant and hence induces a vector field on the reduced manifold $|U|$ whose induced tangent vector at 0 is nonzero. By the classical Frobenius theorem applied to $|U|$, we may assume $X \equiv \partial/\partial z^1 \pmod{\mathcal{J}}$. Thus

$$X = \frac{\partial}{\partial z^1} + \sum_j a_j \frac{\partial}{\partial z^j} + \sum_\rho \beta_\rho \frac{\partial}{\partial \zeta^\rho}.$$

where the a_j and β_ρ are in \mathcal{J} . The a_j have degree zero, hence must lie in \mathcal{J}^2 , and there exists a degree zero matrix of functions $(b_{\rho\tau})$ such that $\beta_\rho \equiv \sum_{\rho, \tau} b_{\rho\tau} \zeta^\tau \pmod{\mathcal{J}^2}$. Hence we have

$$X \equiv \frac{\partial}{\partial z^1} + \sum_{\rho, \tau} b_{\rho\tau} \zeta^\rho \frac{\partial}{\partial \zeta^\tau} \pmod{\mathcal{J}^2}.$$

We now make a transformation $U \rightarrow U$, given by $(z, \zeta) \rightarrow (y, \eta)$, where

$$y = z, \quad \eta = g(z)\zeta,$$

with $g(z) = (g_{\rho\tau}(z))$ an invertible matrix of degree zero functions to be chosen suitably later. A calculation with the chain rule shows that

$$X \equiv \frac{\partial}{\partial y^1} + \sum_\rho \gamma_\rho \frac{\partial}{\partial \eta^\rho} \pmod{\mathcal{J}^2},$$

where $\gamma_\rho = \sum_\tau \zeta^\tau \left(\frac{\partial g_{\rho\tau}}{\partial z^1} + \sum_\sigma g_{\rho\sigma} b_{\sigma\tau} \right)$.

Since antiderivatives exist locally, we may choose g to satisfy the matrix differential equation

$$\begin{aligned}\frac{\partial g}{\partial z^1} &= -gb \\ g(0) &= I.\end{aligned}$$

Then with such a choice of g , g is invertible near 0, (y, η) is a coordinate system near 0 by the \mathbb{Z}_2^n -super inverse function theorem, and

$$X \equiv \frac{\partial}{\partial y^1} \pmod{\mathcal{J}^2}.$$

So we may as well assume that $X \equiv \partial/\partial y^1 \pmod{\mathcal{J}^2}$ from the beginning. Now suppose that $k \geq 2$, and $X \equiv \partial/\partial y_k^1 \pmod{\mathcal{J}^k}$ in some coordinate system (y_k, η_k) . We will show that if we choose a new coordinate system (y_{k+1}, η_{k+1}) given by

$$(y_k, \eta_k) \rightarrow (y_{k+1}, \eta_{k+1}), \quad y_{k+1}^i = y_k^i + a_i^k, \eta_{k+1}^\rho = \eta_k^\rho + \beta_\rho^k$$

with a_i^k, β_ρ^k suitably chosen, then $X \equiv \partial/\partial y_{k+1}^1 \pmod{\mathcal{J}^{k+1}}$. We have:

$$X = \frac{\partial}{\partial y_k^1} + \sum_j g_j^k \frac{\partial}{\partial y_k^j} + \sum_\rho \gamma_\rho^k \frac{\partial}{\partial \eta_k^\rho},$$

where $g_j^k, \gamma_\rho^k \in \mathcal{J}^k$. Assuming for the moment that (y_{k+1}, η_{k+1}) is a coordinate system near 0 (this will be justified later), the chain rule gives

$$\frac{\partial}{\partial y_k^j} = \frac{\partial}{\partial y_{k+1}^j} + \sum_l \left(\frac{\partial a_l^k}{\partial y_k^j} \right) \frac{\partial}{\partial y_{k+1}^l} + \sum_\tau \left(\frac{\partial \beta_\tau^k}{\partial y_k^j} \right) \frac{\partial}{\partial \eta_{k+1}^\tau}$$

and

$$\frac{\partial}{\partial \eta_k^\rho} = \frac{\partial}{\partial \eta_{k+1}^\rho} + \sum_l \left(\frac{\partial a_l^k}{\partial \eta_k^\rho} \right) \frac{\partial}{\partial y_{k+1}^l} + \sum_\tau \left(\frac{\partial \beta_\tau^k}{\partial \eta_k^\rho} \right) \frac{\partial}{\partial \eta_{k+1}^\tau}.$$

As $2k-1 \geq k+1$, we have the following expression for X :

$$X = \frac{\partial}{\partial y_{k+1}^1} + \sum_j \left(g_j^k + \frac{\partial a_j^k}{\partial y_k^1} \right) \frac{\partial}{\partial y_{k+1}^j} + \sum_\rho \left(\gamma_\rho^k + \frac{\partial \beta_\rho^k}{\partial y_k^1} \right) \frac{\partial}{\partial \eta_{k+1}^\rho} + Z,$$

where $Z \equiv 0 \pmod{\mathcal{J}^{k+1}}$. If we choose a_j^k, β_ρ^k such that

$$\begin{aligned}\frac{\partial a_j^k}{\partial y_k^1} &= -g_j^k \\ \frac{\partial \beta_\rho^k}{\partial y_k^1} &= -\gamma_\rho^k,\end{aligned}$$

then $X \equiv \partial/\partial y_{k+1}^1 \pmod{\mathcal{J}^{k+1}}$; such a choice is possible, since antiderivatives exist locally. We note that a_j^k, β_ρ^k so chosen lie in \mathcal{J}^k . This implies the Jacobian matrix of the transformation $(y_k, \eta_k) \mapsto (y_{k+1}, \eta_{k+1})$ at 0 is the identity, hence (y_{k+1}, η_{k+1}) is indeed a coordinate system by the inverse function theorem.

Unlike the ordinary super case, \mathcal{J} is not nilpotent, hence this process does not terminate. We obtain instead an infinite sequence of coordinate systems (y_k, η_k) such that $X \equiv \partial/\partial y_k^1 \pmod{\mathcal{J}^k}$. We must show that (y_k, η_k) converges \mathcal{J} -adically to a unique limit (x, ξ) , that (x, ξ) is a coordinate system near 0, and that $X = \partial/\partial x^1$. For any k , $y_{k+1}^i - y_k^i$ and $\eta_{k+1}^\rho - \eta_k^\rho$ lie in \mathcal{J}^k by the way in which we have defined them, hence $\{y_k^i\}$ and $\{\eta_k^\rho\}$ are \mathcal{J} -adically Cauchy sequences. By Hausdorff completeness of $\mathcal{O}(U)$ [4], they converge to unique limits x^i, ξ^ρ respectively.

As the Jacobian matrix of the morphism $(y_2, \eta_2) \rightarrow (y_k, \eta_k)$ at 0 is the identity for each k by construction of y_k, η_k , the \mathcal{J} -adic continuity of vector fields and Lemma 5.3 together imply that the Jacobian matrix of the morphism $(y_2, \eta_2) \rightarrow (x, \xi)$ at 0 is also the identity, hence (x, ξ) is a coordinate system near 0 by the inverse function theorem.

Finally, let N be any positive integer. There exists some M such that $h_l^k := y_k^l - x^l, \omega_\tau^k := \eta_k^\tau - \xi^\tau \in \mathcal{J}^N$ for all $k \geq M$. By the chain rule, we have

$$\frac{\partial}{\partial y_k^1} = \frac{\partial}{\partial x^1} + \sum_l \left(\frac{\partial h_l^k}{\partial y_k^1} \right) \frac{\partial}{\partial x^l} + \sum_\tau \left(\frac{\partial \omega_\tau^k}{\partial y_k^1} \right) \frac{\partial}{\partial \xi^\tau},$$

which implies $\partial/\partial y_k^1 \equiv \partial/\partial x^1 \pmod{\mathcal{J}^N}$ for all $k \geq M$. Let N' be such that $N' \geq \max(M, N)$. Then $X \equiv \partial/\partial y_{N'}^1 \pmod{\mathcal{J}^{N'}}$ by the definition of the coordinate systems (y_k, η_k) and $\partial/\partial y_{N'}^1 \equiv \partial/\partial x^1 \pmod{\mathcal{J}^N}$ from the previous remark. Hence $X \equiv \partial/\partial x^1 \pmod{\mathcal{J}^N}$ since $\mathcal{J}^{N'} \subseteq \mathcal{J}^N$. But N was arbitrary, hence $X = \partial/\partial x^1$ since $\mathcal{T}M(U)$, being a free $\mathcal{O}(U)$ -module, is $\mathcal{J}(U)$ -adically Hausdorff. \square

Lemma 5.5. *Let \mathcal{D} be a rank $r|0$ distribution locally generated by a set $\{X_1, \dots, X_r\}$ of degree zero \mathbb{Z}_2^r -supercommuting vector fields. Then there exists a coordinate system (x, ξ) such that for each j , $1 \leq j \leq r$,*

$$X_j = \frac{\partial}{\partial x^j} + \sum_{i=1}^{j-1} a_{ij} \frac{\partial}{\partial x^i}$$

where the a_{ij} are degree zero functions.

Proof. Since the statement is local, we may work from the beginning in a coordinate chart U . The proof is by induction on r . The case $r = 1$ is Lem. 5.4. Suppose there exist coordinates (x, ξ) on U having the desired property for $r - 1$ supercommuting degree zero vector fields; we wish to find a coordinate system having the desired property for X_1, \dots, X_r . By hypothesis, $X_j = \frac{\partial}{\partial x^j} + \sum_{i=1}^{j-1} a_{ij} \frac{\partial}{\partial x^i}$ for all $j < r$. We have

$$X_r = \sum_{i=1}^p f_i \frac{\partial}{\partial x^i} + \sum_{\rho=1}^q \varphi_\rho \frac{\partial}{\partial \xi^\rho}$$

where q denotes the number of nonzero-degree coordinates. The hypothesis of supercommutativity implies

$$[X_r, X_j] = \sum_i f_i \left[\frac{\partial}{\partial x^i}, X_j \right] + \sum_\rho \varphi_\rho \left[\frac{\partial}{\partial \xi^\rho}, X_j \right] - \sum_i X_j f_i \frac{\partial}{\partial x^i} - \sum_\rho X_j \varphi_\rho \frac{\partial}{\partial \xi^\rho} = 0.$$

Since $\left[\frac{\partial}{\partial x^i}, X_j \right]$ are a linear combination of the $\frac{\partial}{\partial x^k}$ for $k < r$, we have that $X_j f_i = 0$ for all $i \geq r$. The upper triangular form of the X_j , $j < r$, implies that f_i is a function of $(t^r, \dots, t^p, \xi^1, \dots, \xi^q)$ alone, for $i \geq r$. Furthermore, $\left[\frac{\partial}{\partial \xi^\rho}, X_j \right] = 0$ for all ρ , hence we see that $X_j \varphi_\rho = 0$ for all ρ . By the same reasoning as before, we may conclude that φ_ρ is a function only of $(t^r, \dots, t^p, \xi^1, \dots, \xi^q)$, for all ρ . We may thus write:

$$X_r = \sum_{i=1}^{r-1} f_i \frac{\partial}{\partial x^i} + \sum_{k=r}^p f_k \frac{\partial}{\partial x^k} + \sum_{\rho=1}^q \varphi_\rho \frac{\partial}{\partial \xi^\rho}$$

Let $X'_r := \sum_{k=r+1}^p f_k \frac{\partial}{\partial x^k} + \sum_{\rho=1}^q \varphi_\rho \frac{\partial}{\partial \xi^\rho}$. Note that X'_r depends only on the coordinates $x^r, \dots, x^p, \xi^1, \dots, \xi^q$. Since X_r is \mathcal{O} -linearly independent of X_1, \dots, X_{r-1} , the upper triangular form of the X_j for $j < r$ and Prop. 5.4 applied to X'_r imply that we may change the coordinates $x^r, \dots, x^p, \xi^1, \dots, \xi^q$ so that there is a new coordinate system $(x^1, \dots, x^{r-1}, x'^r, \dots, x'^p, \xi')$ on U (shrinking U if needed) such that

$$X_r = \frac{\partial}{\partial x'^r} + \sum_{i=1}^{r-1} f_i \frac{\partial}{\partial x^i},$$

as desired. It is clear that the upper triangular form of the X_j , $j \leq r-1$, remains unchanged when the X_j are expressed in the new coordinate system. \square

The preceding discussion suffices to prove the local Frobenius theorem for involutive distributions of degree zero. For the general case, we begin by showing the existence of a canonical form for a local, nondegenerate vector field of nonzero degree.

Proposition 5.6. *Let χ be a vector field on M of nonzero degree defined in a neighborhood of m , such that $\chi_m \neq 0$. If χ is odd, suppose in addition that $\chi^2 = 0$. Then there is a coordinate system (x, ξ) about m such that $\chi = \partial/\partial \xi^\sigma$, where the degree of ξ^σ is equal to that of χ .*

Proof. Since the proposition is local, we may assume that $M = U \subseteq \mathbb{R}^{p|\mathbf{q}}$ with coordinates (z, ζ) such that m corresponds to 0, in which case $\mathcal{T}M(U)$ is a free $\mathcal{O}(U)$ -module. Then χ has the form

$$\chi = \sum_j \alpha_j(z, \zeta) \frac{\partial}{\partial z^j} + \sum_\rho a_\rho(z, \zeta) \frac{\partial}{\partial \zeta^\rho}.$$

Without loss of generality, the condition $\chi_m \neq 0$ may be taken to be $a_\sigma(0,0) \neq 0$ for some index σ of degree $\deg(\chi)$. By reordering the coordinates ζ^ρ such that $\deg(\zeta^\rho) = \deg(\chi)$, we may assume that ζ^σ is the first coordinate of degree $\deg(\chi)$.

Let $0|1_\sigma$ denote the \mathbb{Z}_2^n -superdimension which is 0 in all degrees except $\deg(\sigma)$ and 1 in degree $\deg(\sigma)$, and $p|\widehat{\mathbf{q}}$ the codimension of $0|1_\sigma$ in $p|\mathbf{q}$. Let η^σ be a coordinate on $\mathbb{R}^{0|1_\sigma}$. We define a morphism $\mathbb{R}^{0|1_\sigma} \times U^{p|\widehat{\mathbf{q}}} \rightarrow U^{p|\mathbf{q}}$ by

$$\begin{aligned} z^j &= y^j + \theta^\sigma \alpha_j(y, 0, \widehat{\eta}) \\ \zeta^\sigma &= \eta^\sigma a_\sigma(y, 0, \widehat{\eta}) \\ \zeta^\rho &= \eta^\rho + \eta^\sigma a_\rho(y, 0, \widehat{\eta}), \quad \rho \neq \sigma, \end{aligned}$$

where by convention $\alpha_j(y, 0, \widehat{\eta})$ (resp. $a_\rho(y, 0, \widehat{\eta})$) denotes $\alpha_j(y, \eta)|_{\eta^\sigma=0}$ (resp. $a_\rho(x, \eta)|_{\eta^\sigma=0}$). Note that these are functions of y and of η^ρ for all $\rho \neq \sigma$. The Jacobian matrix of this morphism at 0 has the form:

$$\begin{pmatrix} I_{p'} & * & 0 \\ 0 & a_\sigma(0) & 0 \\ 0 & * & I_{q'} \end{pmatrix}$$

which is invertible, hence the morphism $(y, \eta) \mapsto (z, \zeta)$ may be regarded as a change of coordinates near 0 by the inverse function theorem. From the chain rule, we have

$$\frac{\partial}{\partial \eta^\sigma} = \sum_j \alpha_j(y, 0, \widehat{\eta}) \frac{\partial}{\partial z^j} + \sum_\rho a_\rho(y, 0, \widehat{\eta}) \frac{\partial}{\partial \zeta^\rho}.$$

We must express the coefficient functions as functions of the new coordinates z^j, ζ^ρ . For this, note that by the Taylor series expansion of \mathbb{Z}_2^n -superfunctions,

$$\begin{aligned} \alpha_j(z, \zeta) &= \alpha_j(y^i + \eta^\sigma \alpha_i, \eta^\sigma a_\sigma, \eta^{\rho \neq \sigma} + \eta^\sigma a_{\rho \neq \sigma}) \\ &= \alpha_j(y, 0, \widehat{\eta}) + \eta^\sigma \kappa \end{aligned}$$

for some function κ of degree 0. Similarly, we find $a_\rho(z, \zeta) = a_\rho(y, 0, \widehat{\eta}) + \eta^\sigma h$ for some function h of degree $\deg(\chi)$. Hence we have

$$\chi = \frac{\partial}{\partial \eta^\sigma} + \eta^\sigma V,$$

where V is a vector field of degree zero. The proof now splits into two cases according to the parity of χ .

χ even. The argument mirrors that of Prop 5.4. From the preceding discussion, there exists a coordinate system (y, η) such that $\chi \equiv \partial/\partial \eta^\sigma \pmod{\mathcal{J}}$. Now suppose that $k \geq 1$, and that $\chi \equiv \partial/\partial \eta_k^\sigma \pmod{\mathcal{J}^k}$ in some coordinate system (y_k, η_k) . We will show that if we choose a new coordinate system (y_{k+1}, η_{k+1}) given by

$$(y_k, \eta_k) \rightarrow (y_{k+1}, \eta_{k+1}), \quad y_{k+1}^i = y_k^i + a_i^k, \eta_{k+1}^\rho = \eta_k^\rho + \beta_\rho^k$$

with a_i^k, β_ρ^k suitably chosen, then $\chi \equiv \partial/\partial\eta_{k+1}^\sigma \pmod{\mathcal{J}^{k+1}}$. We have

$$(5.3) \quad \chi \equiv \frac{\partial}{\partial\eta_k^\sigma} + \sum_j \gamma_j^k \frac{\partial}{\partial y_k^j} + \sum_\rho g_\rho^k \frac{\partial}{\partial\eta_k^\rho}$$

where $\gamma_j^k, g_\rho^k \in \mathcal{J}^k$. Under the assumption that (y_{k+1}, η_{k+1}) is a coordinate system (which will be justified later), the chain rule again yields

$$\frac{\partial}{\partial y_k^j} = \frac{\partial}{\partial y_{k+1}^j} + \sum_l \left(\frac{\partial a_l^k}{\partial y_k^j} \right) \frac{\partial}{\partial y_{k+1}^l} + \sum_\tau \left(\frac{\partial \beta_\tau^k}{\partial y_k^j} \right) \frac{\partial}{\partial \eta_{k+1}^\tau}$$

and

$$\frac{\partial}{\partial\eta_k^\rho} = \frac{\partial}{\partial\eta_{k+1}^\rho} + \sum_l \left(\frac{\partial a_l^k}{\partial\eta_k^\rho} \right) \frac{\partial}{\partial y_{k+1}^l} + \sum_\tau \left(\frac{\partial \beta_\tau^k}{\partial\eta_k^\rho} \right) \frac{\partial}{\partial\eta_{k+1}^\tau}.$$

Substituting, we have the following expression for χ :

$$\chi = \frac{\partial}{\partial\eta_{k+1}^\sigma} + \sum_j \left(\gamma_j^k + \frac{\partial a_j^k}{\partial\eta_k^\sigma} \right) \frac{\partial}{\partial y_{k+1}^j} + \sum_\rho \left(g_\rho^k + \frac{\partial \beta_\rho^k}{\partial\eta_k^\sigma} \right) \frac{\partial}{\partial\eta_{k+1}^\rho} + Z,$$

where Z denotes the remaining terms (these will be shown to be zero mod \mathcal{J}^{k+1} once a_j^k, β_ρ^k are suitably chosen). We now choose a_j^k, β_ρ^k such that:

$$\begin{aligned} \frac{\partial a_j^k}{\partial\eta_k^\sigma} &= -\gamma_j^k \text{ and } \beta_j^k(y, 0, \widehat{\eta}) = 0; \\ \frac{\partial \beta_\rho^k}{\partial\eta_k^\sigma} &= -g_\rho^k \text{ and } a_\rho^k(y, 0, \widehat{\eta}) = 0, \end{aligned}$$

where the notational convention regarding the meaning of $\beta_j^k(y, 0, \widehat{\theta}), a_\rho^k(y, 0, \widehat{\theta})$ continues to hold. Note that the latter conditions ensure that a_j^k and β_ρ^k lie in \mathcal{J}^{k+1} . Hence one sees that for such a choice of a_j^k and β_ρ^k , $Z \equiv 0 \pmod{\mathcal{J}^{k+1}}$ and so $\chi \equiv \partial/\partial\eta_{k+1}^\sigma \pmod{\mathcal{J}^{k+1}}$ as well. Choosing a_j^k and β_ρ^k satisfying these conditions is clearly possible if the γ_j^k (resp. g_ρ^k) are polynomial in θ_k^σ , and by Hausdorff completeness of \mathcal{O} and \mathcal{J} -adic continuity of vector fields it is possible for arbitrary γ_j^k (resp. g_ρ^k). The Jacobian matrix of the transformation $(y_k, \eta_k) \mapsto (y_{k+1}, \eta_{k+1})$ at 0 is clearly the identity and so (y_{k+1}, η_{k+1}) so defined is a coordinate system by the inverse function theorem, as previously asserted.

As in Prop. 5.4, we obtain an infinite sequence of coordinate systems (y_k, η_k) such that $\chi \equiv \partial/\partial\eta_k^1 \pmod{\mathcal{J}^k}$. For any k , $\{y_k^i\}$ and $\{\eta_k^\rho\}$ are \mathcal{J} -adically Cauchy sequences by construction, hence converge to unique limits x^i, ξ^ρ respectively. The Jacobian matrix of the morphism $(y_1, \eta_1) \rightarrow (y_k, \eta_k)$ at 0 is the identity for any k by construction of y_k, η_k , so the Jacobian matrix of the morphism $(y_1, \eta_1) \rightarrow (x, \xi)$ at 0 is also the identity and (x, ξ) is a coordinate system near 0.

Finally, let N be any positive integer. There exists some M such that $h_l^k := y_k^l - x^l, \omega_\tau^k := \eta_k^\tau - \xi^\tau \in \mathcal{J}^{N+1}$ for all $k \geq M$. By the chain rule, we have

$$\frac{\partial}{\partial \eta_k^\sigma} = \frac{\partial}{\partial \xi^\sigma} + \sum_l \left(\frac{\partial h_l^k}{\partial \eta_k^\sigma} \right) \frac{\partial}{\partial x^l} + \sum_\tau \left(\frac{\partial \omega_\tau^k}{\partial \eta_k^\sigma} \right) \frac{\partial}{\partial \xi^\tau},$$

which implies $\partial/\partial \eta_k^\sigma \equiv \partial/\partial \xi^\sigma \pmod{\mathcal{J}^N}$ for all $k \geq M$. Let N' be such that $N' \geq \max(M, N)$. Then $\chi \equiv \partial/\partial \eta_{N'}^\sigma \pmod{\mathcal{J}^{N'}}$ by the definition of the coordinate systems (x_k, θ_k) and $\partial/\partial \eta_{N'}^\sigma \equiv \partial/\partial \xi^\sigma \pmod{\mathcal{J}^N}$ from the previous remark. Hence $\chi \equiv \partial/\partial \xi^\sigma \pmod{\mathcal{J}^N}$. But N was arbitrary, hence $\chi = \partial/\partial \xi^\sigma$ since $\mathcal{T}M(U)$ is $\mathcal{J}(U)$ -adically Hausdorff.

χ odd. In this case $(\eta^\sigma)^2 = 0$, and so we may conclude as in the \mathbb{Z}_2 -graded case by using the hypothesis that $\chi^2 = 0$. Namely, the equation $\chi = \partial/\partial \eta^\sigma + \eta^\sigma V$ then implies

$$\chi^2 = V - \eta^\sigma W = 0$$

for some vector field W of degree η^σ , which implies $\chi = \partial/\partial \eta^\sigma$. \square

Now, we may finally prove the more difficult half of the local Frobenius theorem:

Theorem 5.7. *Let \mathcal{D} be an involutive distribution of rank $r|\mathbf{s}$ on a \mathbb{Z}_2^n -supermanifold M . Then for any point m of M , there exist local coordinates (x, ξ) about m such that*

$$\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^r}, \frac{\partial}{\partial \xi^1}, \dots, \frac{\partial}{\partial \xi^s}$$

form a local \mathcal{O} -module basis of \mathcal{D} .

Proof. Let $\{X_1, \dots, X_r, \chi_1, \dots, \chi_s\}$ be a local basis of \mathcal{D} . By Prop. 5.2, we may assume that these vector fields \mathbb{Z}_2^n -supercommute. Hence, $\mathcal{D}' := \text{span}\{X_1, \dots, X_r\}$ is an integral subdistribution of rank $r|0$ and by Lem. 5.5, there exist local coordinates such that $X_i = \frac{\partial}{\partial x^i}$.

Consider the representation of χ_1 in terms of the coordinate vector fields:

$$\chi_1 = \sum_j \beta_1^j \frac{\partial}{\partial x^j} + \sum_\rho b_1^\rho \frac{\partial}{\partial \xi^\rho}.$$

Then from the fact that $[X_i, \chi_1] = 0$ for all i , we conclude that the coefficients β_1^j, b_1^ρ depend only on $x^{r+1}, \dots, x^p, \xi^1, \dots, \xi^q$. The representation of χ_1 may contain multiples of the $\frac{\partial}{\partial x^j}$, $j = 1, \dots, r$. However, we may eliminate these terms by subtracting off suitable linear combinations of X_1, \dots, X_r .

We may therefore assume that χ_1 depends only on the coordinates $x^{r+1}, \dots, x^p, \xi^1, \dots, \xi^q$. We note that if χ_1 is odd, $\chi_1^2 = 0$ by the supercommutativity of the vector fields. Hence we may change the coordinates $x^{r+1}, \dots, x^p, \xi^1, \dots, \xi^q$ by Lem. 5.6 in such a way that $\chi_1 = \frac{\partial}{\partial \xi^1}$.

For χ_2 , we apply the same idea: the fact that $[\chi_2, X_i] = 0$ and $[\chi_2, \chi_1] = 0$ implies that χ_2 depends only on $x^{r+1}, \dots, x^p, \xi^2, \dots, \xi^q$ and we may again apply Lem. 5.6 to find a new coordinate system in which $\chi_2 = \frac{\partial}{\partial \xi^2}$. We can clearly apply this argument successively to χ_3, \dots, χ_s to conclude the proof. \square

6. SUBMANIFOLDS

The following definitions are straightforward extensions of the \mathbb{Z}_2 -case. As in the ungraded case, one must distinguish between immersed and embedded submanifolds. Note that we omit prefixes such as “ \mathbb{Z}_2^n -super-,” etc. in order to make the terminology less cumbersome.

Definition 5. Let M be a \mathbb{Z}_2^n -supermanifold. An *immersed submanifold* of M is a pair (N, j) where N is a \mathbb{Z}_2^n -supermanifold and $j : N \rightarrow M$ is an injective immersion; that is, $|j| : |N| \rightarrow |M|$ is injective and $(dj)_m$ is injective at each point m of M . An *embedding* is a morphism $j : N \rightarrow M$ such that j is an immersion and $|j|$ is a homeomorphism of the topological space $|N|$ onto its image in $|M|$.

A pair (N, j) is an *embedded submanifold* of M if j is an embedding. (N, j) is a *closed embedded submanifold* if j is an embedding and $|j|(|N|)$ is a closed subset of $|M|$.

We will just need the following simple property of immersions for our proof of the global Frobenius theorem.

Proposition 6.1. *Let $j : M \rightarrow N$ be an immersion. Then j is locally an embedding on M : for any point $m \in |M|$, there is an open neighborhood $U \ni m$ such that $j|_U$ is an embedding.*

Proof. This is a consequence of the discussion of local forms of morphisms in [6]: since j is an immersion, at $m \in |M|$ there exist charts U around m and V around $j(m)$ such that j is a linear embedding. \square

7. GLOBAL \mathbb{Z}_2^n -SUPER FROBENIUS THEOREM

We now turn our attention to the global version of the \mathbb{Z}_2^n -super Frobenius theorem.

Definition 6. Let \mathcal{D} be a distribution on M . An *integral submanifold* of M is an immersed submanifold $j : N \rightarrow M$ such that for each $n \in N$, $(dj_n)(T_n N) = D_{j(n)}$. An integral submanifold $j : N \rightarrow M$ is *maximal* if $|N|$ is connected and N contains every other connected integral submanifold of M which has a point in common with it.

We will say is a coordinate system (x, ξ) on a domain U is *adapted* to \mathcal{D} if $\{\partial/\partial x^i, \partial/\partial \xi^j\}$, $1 \leq i \leq r, 1 \leq j \leq s$ is a basis for $\mathcal{D}(U)$ as $\mathcal{O}(U)$ -module.

The following lemma will allow us to construct maximal integral submanifolds.

Lemma 7.1. *Let $\{N_i\}_{i \in I}$ be any nonempty collection of connected integral submanifolds of \mathcal{D} passing through a point p . Then there is a unique structure of \mathbb{Z}_2^n -supermanifold N on $|N| := \cup_i |N_i|$ such that N is an connected integral submanifold of \mathcal{D} for which the inclusion $N_i \rightarrow N$ is an open embedding for each i .*

Proof. We define a topology on $\cup_i |N_i|$ by declaring a set $U \subset \cup_{i \in I} |N_i|$ to be open if and only if $U \cap |N_i|$ is open in $|N_i|$ for all i . Noting that the $|N_i|$ are integral submanifolds for the reduced distribution $\tilde{\mathcal{D}}$ on $|M|$, it follows that each $|N_i|$ is an open subset of $|N|$ and that $|N|$ is Hausdorff and second-countable via the same arguments as in the classical case.

It remains to endow $|N|$ with the structure of a \mathbb{Z}_2^n -supermanifold. For this, let us take the \mathbb{Z}_2^n -supermanifold atlas defined by all charts $(T \cap N, \varphi)$, where T is a single slice in an adapted chart U for \mathcal{D} , and $\varphi : T \rightarrow \mathbb{R}^{r|s}$ is the morphism which is given in the adapted coordinates by linear projection: $\varphi(x^1, \dots, x^r, \dots, x^p, \xi^1, \dots, \xi^s, \dots, \xi^q) = (x^1, \dots, x^r, \xi^1, \dots, \xi^s)$. (Such a chart exists around any point of N by the local Frobenius theorem 5.1). Any slice T is a closed embedded submanifold of U .

Let T, T' be two such slices in adapted charts U, U' . Then $V := U \cap U'$ is also an adapted chart, and $T \cap V, T' \cap V$ are embedded integral submanifolds of \mathcal{D} in V . Suppose that $|T \cap N|$ and $|T' \cap N|$ intersect. Then a standard lemma on the local structure of integral manifolds for a distribution on an ungraded manifold, applied to $\tilde{\mathcal{D}}$, implies $|T \cap V|, |T' \cap V|$ are contained in a single slice \tilde{S} for $\tilde{\mathcal{D}}$ inside $|V|$, and

since all integral submanifolds for $\tilde{\mathcal{D}}$ are equidimensional, the inclusions $|T \cap V| \rightarrow \tilde{S}, |T' \cap V| \rightarrow \tilde{S}$ are open embeddings, whence $T \cap T'$ is an open subsupermanifold of T and T' .

As T, T' are embedded, the transition maps $\varphi' \circ \varphi^{-1}$ are isomorphisms, showing that the atlas defines a \mathbb{Z}_2^n -supermanifold structure on $|N|$. This implies that the inclusion $j : N \rightarrow M$ is an immersion (since j is locally an embedding). Furthermore $dj_n(T_n N) = D_{j(n)}$ at any point n , since this is true on any slice in an adapted chart around n , so that N is an integral submanifold. Suppose N' is another \mathbb{Z}_2^n -supermanifold structure on $|N|$ such that N' is an integral submanifold. Then $i' : N' \rightarrow M$ is an immersion, so locally an embedding. Hence the \mathbb{Z}_2^n -supermanifold structure on N' must match that of the slices on sufficiently small open subsets of $|N|$. This proves the desired uniqueness. \square

Theorem 7.2 (Global Frobenius theorem for \mathbb{Z}_2^n -supermanifolds). *Let M be a \mathbb{Z}_2^n -supermanifold and \mathcal{D} an involutive distribution on M . Then given any point m of M , there exists a unique maximal integral subsupermanifold of \mathcal{D} passing through m .*

Proof. By the local Frobenius theorem, the collection $\{N_i\}$ of all connected integral submanifolds of \mathcal{D} passing through m is nonempty. By Lem. 7.1, $\{N_i\}$ determines a unique connected integral submanifold N_m passing through m . N_m is clearly maximal since any other connected integral submanifold passing through m belongs to $\{N_i\}$, so must be contained in N_m . Uniqueness follows for the same reason: let N'_m be another maximal integral submanifold through m . Then N'_m belongs to $\{N_i\}$ and hence is contained in N_m . But then $N'_m = N_m$ by maximality. \square

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